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ON p -ADIC COMPACT OPERATORS

by

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ABSTRACT. A basic theory for compact and (closely related) semi Fredholm operators on p -adic Banach spaces is given.

INTRODUCTION. Compact operators on Banach spaces E over a non-archimedean valued field K have been studied in [2], [4], [5], [6], [7], [11]. In some of these, severe restrictions have been imposed on K or E (viz. K spherically complete or E has an orthogonal base). Also, the techniques used in these various papers differ. The purpose of this paper is to set up a theory of compact operators by using functional analytic methods, yielding also many new results. In appropriate sections the connection to the existing literature is explained.

TERMINOLOGY. Throughout K is a non-archimedean valued field that is complete under the metric induced by its non-trivial valuation $|\cdot|$, and E, F, G, \dots are K -Banach spaces with a (non-archimedean) norm denoted $\|\cdot\|$. With the usual operator norm, $\mathcal{L}(E, F)$ is the K -Banach space consisting of all continuous linear maps: $E \rightarrow F$ and $\mathcal{L}(E) := \mathcal{L}(E, E)$. The *dual space* of E is $E' := \mathcal{L}(E, K)$. The *adjoint* $A' \in \mathcal{L}(F', E')$ of an $A \in \mathcal{L}(E, F)$ is defined as usual. The closed unit ball $\{x \in E : \|x\| \leq 1\}$ of E is denoted by B_E . The (norm) closure of a set $X \subset E$ is \overline{X} .

A nonempty subset A of E is *absolutely convex* if $x, y \in A$, $\lambda, \mu \in K$, $|\lambda| \leq 1$, $|\mu| \leq 1$ implies $\lambda x + \mu y \in A$. The smallest absolutely convex set containing $X \subset E$ is $\text{co}X$, the K -linear hull of X is $[X]$. We shall write $\overline{\text{co}X}$ instead of $\overline{\text{co}X}$.

A subset X of E is a *compactoid* (*local compactoid*) if for each zero neighbourhood U in E there exists a finite set $F \subset E$ such that $X \subset U + \text{co}F$ ($X \subset U + [F]$).

A map $T \in \mathcal{L}(E, F)$ is *compact* if $T(B_E)$ is a compactoid. A map $A \in \mathcal{L}(E, F)$ is *semi-Fredholm* if $\text{Ker}A$ is finite dimensional and $\text{Im}A$ is closed.

For unexplained terms and background we refer to [9], [15], [12] (locally convex and Banach spaces), [9], [10], [13] ((local) compactoids and compact operators), [7] (semi-Fredholm operators).

§ 1. COMPLETE CONTINUITY

For a $T \in \mathcal{L}(E, F)$ let us introduce property (*):

(*) T maps weakly convergent sequences into norm convergent sequences.

The relation between 'compactness' and (*) differs from the Archimedean theory:

PROPOSITION 1.1. *Let K be spherically complete. Then each $T \in \mathcal{L}(E, F)$ has property (*).*

Proof. By [9], Theorem 4.55 each weakly convergent sequence is norm convergent.

Remark. The conclusion of Proposition 1.1 holds also if K is not spherically complete and E is a strongly polar Banach space (e.g. c_0), see [12]. However, if $E = l^\infty$ and K is not spherically complete then for $T \in \mathcal{L}(l^\infty \rightarrow F)$, property (*) implies compactness of T (for the converse, see Theorem 1.2). *Proof.* By reflexivity ([9], Theorem 4.17) the sequence $l_1 := (1, 0, 0, \dots)$, $l_2 := (0, 1, 0, \dots)$, $l_3 := (0, 0, 1, 0, \dots)$ converges weakly to 0, so $A := \overline{\text{co}}\{Tl_1, Tl_2, \dots\}$ is a compactoid in F . Let $x \in B_{l^\infty}$; we prove that $Tx \in A$. In fact, set $x = (\xi_1, \xi_2, \dots)$, $x_n = (\xi_1, \xi_2, \dots, \xi_n, 0, 0, \dots)$. It is easily seen that $\lim_{n \rightarrow \infty} (x - x_n) = 0$ weakly so that $Tx = \lim_{n \rightarrow \infty} Tx_n \in A$.

PROBLEM. For nonspherically complete K , study the class of operators T in $\mathcal{L}(E, F)$ that have property (*).

On the other hand we have the following theorem that has a well known Archimedean counterpart (see [1], Theorem VI 5.6). Notice that we do not require the weak topology on E to be Hausdorff!

THEOREM 1.2. *Let $\mathcal{L}(E, F)$. The following are equivalent.*

(α) T is compact.

(β) If $(x_i)_{i \in I}$ is a bounded net converging weakly to 0 then $(Tx_i)_{i \in I}$ converges to 0 in the norm topology.

Proof. (α) \implies (β). By continuity of T we have $Tx_i \rightarrow 0$ weakly in $D := \overline{TE}$. By compactness of T , D is of countable type (hence, a polar space) and $X := \{Tx_i : i \in I\}$ is a norm compactoid in D . By [12], Theorem 5.12 the norm topology and the weak topology of D coincide on $X \cup \{0\}$. It follows that $Tx_i \rightarrow 0$ in the norm sense.

(β) \implies (α). We shall prove that, if D is a subspace of E for which $\inf\{\|Tx\|/\|x\| : x \in D, x \neq 0\} > 0$ then $\dim D < \infty$ (compactness of T then follows from [9], Theorem 4.40 (η) \implies (α)). In fact, let (x_i) be a net in the unit ball of D converging to 0 in the weak topology of D . Then also $x_i \rightarrow 0$ in the weak topology of E . By (β), $Tx_i \rightarrow 0$ in the norm of TD so, by assumption, $x_i \rightarrow 0$ in the norm topology of D . Thus, the unit

ball of D , being a compactoid for the weak topology of D , is also a compactoid for the norm topology. It follows that $\dim D < \infty$.

§ 2. BASES (e_i) FOR WHICH $Te_i \rightarrow 0$

If T is a compact operator on a separable Hilbert space H over \mathbb{C} then for any orthonormal base e_1, e_2, \dots of H we have $\lim_{n \rightarrow \infty} Te_n = 0$ (since $\lim_{n \rightarrow \infty} e_n = 0$ weakly). Although in the p -adic theory the standard base e_1, e_2, \dots of c_0 is not weakly convergent the question as to whether $\lim_{n \rightarrow \infty} Te_n = 0$ for some (all) compact $T \in \mathcal{L}(c_0)$ makes good sense. Our first Proposition reveals a strong deviation from the classical theory.

PROPOSITION 2.1. *Let $T \in \mathcal{L}(c_0)$ be such that $\lim_{n \rightarrow \infty} Te_n = 0$ for each orthonormal base e_1, e_2, \dots of c_0 . Then $T = 0$.*

Proof. Let $x \in c_0$, $x \neq 0$. To prove $Tx = 0$ we may assume $\|x\| = 1$. By [9], Lemma 4.35 and Theorem 5.9 there exist $e_1, e_2, \dots \in c_0$ such that x, e_1, e_2, \dots is an orthonormal base of c_0 . Then $\lim_{n \rightarrow \infty} Te_n = 0$. But also $x, x - e_1, x - e_2, \dots$ is an orthonormal base of c_0 so that $\lim_{n \rightarrow \infty} T(x - e_n) = 0$ i.e. $Tx = 0$.

On the other hand, for any fixed orthonormal base e_1, e_2, \dots of c_0 the set $\{T \in \mathcal{L}(c_0) : \lim_{n \rightarrow \infty} Te_n = 0\}$ is nontrivial and consists only of compact operators. So one might ask whether for each compact $T \in \mathcal{L}(c_0)$ there exists an orthonormal base e_1, e_2, \dots (depending on T) for which $\lim_{n \rightarrow \infty} Te_n = 0$. To obtain the answer (given in Corollary 2.5) we shall first allow the base to be t -orthogonal.

LEMMA 2.2. *Let E be of countable type, let $T \in \mathcal{L}(E, F)$ be compact. Let $x \in E$, $t \in (0, 1)$, $\varepsilon > 0$. Then E has a t -orthogonal decomposition $E = D \oplus H$ where D and H are closed subspaces such that*

- (i) $\dim D < \infty$, $x \in D$,
- (ii) $\|T|H\| < \varepsilon$.

Proof. Kx has a \sqrt{t} -orthogonal complement M . Since $T|M$ is compact there is a closed subspace $H \subset M$ of finite codimension such that $\|T|H\| < \varepsilon$ ([9] Theorem 4.40). H has a \sqrt{t} -orthogonal complement M_1 in M . Set $D := Kx + M_1$. It is easily seen that D and H are t -orthogonal.

THEOREM 2.3. *Let E be an infinite-dimensional Banach space of countable type. Then for each compact $T \in \mathcal{L}(E, F)$ and each $t \in (0, 1)$ there exists a t -orthogonal base e_1, e_2, \dots of E with $\inf_n \|e_n\| > 0$ and $\lim_{n \rightarrow \infty} Te_n = 0$.*

Proof. Let $b_1, b_2, \dots \in E$ be such that $[b_1, b_2, \dots]$ is dense in E . Choose $t_1, t_2, \dots \in (0, 1)$ such that $\prod_n t_n = \sqrt{t}$. According to Lemma 2.2 we can make a t_1 -orthogonal decomposition $E = D_1 \oplus H_1$ where $\dim D_1 < \infty$, $b_1 \in D_1$ and $\|T|H_1\| \leq \frac{1}{2}$. Write $b_2 = d_1 + h_1$ where $d_1 \in D_1$, $h_1 \in H_1$. By Lemma 2.2 we can make a t_2 -orthogonal decomposition $H_1 = D_2 \oplus H_2$ where $\dim D_2 < \infty$, $h_1 \in D_2$ and $\|T|H_2\| \leq \frac{1}{4}$ etc.

Inductively we arrive at a sequence of finite dimensional spaces D_1, D_2, \dots such that for each $n \in \mathbb{N}$

- (i) $\{b_1, b_2, \dots, b_n\} \subset D_1 + D_2 + \dots + D_n$
- (ii) $\|T|_{D_{n+1}}\| \leq 2^{-n}$
- (iii) D_n and $D_{n+1} + D_{n+2} + \dots$ are t_n -orthogonal.

From (iii) it follows that D_1, D_2, \dots is a $\Pi t_n = \sqrt{t}$ -orthogonal sequence of subspaces. By choosing in every nonzero D_n a \sqrt{t} -orthogonal base we obtain a t -orthogonal sequence e_1, e_2, \dots and we can arrange that $\inf_n \|e_n\| > 0$ and $\sup_n \|e_n\| < \infty$. Then (i) implies that e_1, e_2, \dots is a base of E , while from (ii) it follows that $\lim_{n \rightarrow \infty} Te_n = 0$.

Remark. There is no ‘obvious’ version of Theorem 2.3 for spaces E that are *not* of countable type! In fact, here is a space E with an uncountable orthonormal base and a compact $T : E \rightarrow c_0$ such that for each $t \in (0, 1)$ and each t -orthogonal base (e_i) of E with $\inf \|e_i\| > 0$ we do not have $\lim Te_i = 0$: Choose a set I with $\#I \geq \#l^\infty$ and make a continuous linear surjection $\pi : E := c_0(I) \rightarrow l^\infty$, let $S : l^\infty \rightarrow c_0$ be any injective compact operator. Then $T := S\pi$ is compact. If for some t -orthogonal base $(e_i)_{i \in I}$ of E with $\inf_i \|e_i\| > 0$ we had $\lim_i Te_i = 0$ then $\{i \in I; \pi e_i \neq 0\}$ is countable implying $l^\infty \simeq E/\text{Ker } \pi$ is of countable type, a contradiction.

Next we discuss, given a compact operator T on a space of countable type, the existence of an *orthogonal* base e_1, e_2, \dots for which $\inf_n \|e_n\| > 0$ and $\lim_{n \rightarrow \infty} Te_n = 0$.

THEOREM 2.4. *For an infinite-dimensional Banach space E of countable type the following are equivalent.*

- (α) *For any Banach space F and each compact $T \in \mathcal{L}(E, F)$ there exists an orthogonal base e_1, e_2, \dots of E with $\inf_n \|e_n\| > 0$ and $\lim_{n \rightarrow \infty} Te_n = 0$.*
- (β) *For each $f \in E'$ there exists an orthogonal base e_1, e_2, \dots of E with $\inf_n \|e_n\| > 0$ and $\lim_{n \rightarrow \infty} f(e_n) = 0$.*
- (γ) *For each $f \in E'$ there exists an $x \in E$, $x \neq 0$ with $|f(x)| = \|f\| \|x\|$.*
- (δ) *Every strictly decreasing sequence of values of the norm function converges to 0.*
- (ϵ) *The valuation of K is discrete. E is spherically complete.*

Proof. (α) \implies (β) is trivial, (β) \implies (γ) is easy (as $n \mapsto |f(e_n)|/\|e_n\|$ is a null sequence, $\|f\| = \sup_n \frac{|f(e_n)|}{\|e_n\|} = \max_n \frac{|f(e_n)|}{\|e_n\|}$).

(γ) \implies (δ). Let $\{e_i : i \in \mathbb{N}\}$ be a maximal orthogonal subset of E ; we prove it to be a base of E . Suppose $D := \overline{\{e_i : i \in \mathbb{N}\}} \neq E$. Then choose an $f \in E'$, $f \neq 0$, $f(D) = \{0\}$. By (γ) there is an $x \in E$, $x \neq 0$ with $|f(x)| = \|f\| \|x\|$. For each $d \in D$ we have

$$\|f\| \|x\| = |f(x - d)| \leq \|f\| \|x - d\|$$

implying $x \perp D$, a contradiction. Now (δ) follows from [9] Theorems 5.13 and 5.16 and so does $(\delta) \iff (\varepsilon)$. Finally, we prove $(\varepsilon) \implies (\alpha)$. From [9], Theorems 5.13 and 5.16 it follows that if $D_1 \subset D_2 \subset E$ are closed subspaces then D_1 has an orthocomplement in D_2 . To arrive at (α) read the proofs of Lemma 2.2 and Theorem 2.3 for $t = t_n = 1$ for all $n \in \mathbb{N}$.

COROLLARY 2.5.

- (i) *For each compact $T \in \mathcal{L}(c_0)$ and $t \in (0, 1)$ there exists a t -orthogonal base e_1, e_2, \dots of c_0 with $\|e_n\| = 1$ for all n and $\lim_{n \rightarrow \infty} Te_n = 0$.*
- (ii) *There exists for each compact $T \in \mathcal{L}(c_0)$ an orthonormal base e_1, e_2, \dots of c_0 with $\lim_{n \rightarrow \infty} Te_n = 0$ if and only if the valuation of K is discrete.*

§ 3. SEMI-FREDHOLM AND COMPACT OPERATORS

The main results of this section (Theorems 3.1 and 3.4) were proved in [8] in the course of characterizing semi-Fredholm operators by preservation of orthogonality. Here we have a different approach.

Throughout § 3 we shall denote the natural factorization of an $A \in \mathcal{L}(E, F)$ as follows:

$$\begin{array}{ccc} E & \xrightarrow{A} & F \\ \pi_A \searrow & & \nearrow A_1 \\ & E/\text{Ker } A & \end{array}$$

where $E/\text{Ker } A$ has the quotient norm.

THEOREM 3.1. ([8], Corollary 2.6) *Let $A \in \mathcal{L}(E, F)$. Then the following are equivalent.*

(α) *A is semi-Fredholm.*

(β) *If X is a local compactoid in F then $A^{-1}(X)$ is a local compactoid in E .*

Proof. (α) \implies (β). We may assume that X is absolutely convex and closed. As $\text{Im } A_1 = \text{Im } A$ is closed we have by the open mapping theorem that A_1 is a linear homomorphism onto $\text{Im } A_1$. Then $A_1^{-1}(X)$, being isomorphic to $X \cap \text{Im } A_1$ is a closed local compactoid. By [10], Corollary 6.5 we have $A_1^{-1}(X) = D \oplus Y$ where D is a finite dimensional vector space and where Y is an absolutely convex closed compactoid. From [3], Proposition 2.5 it follows that there is a compactoid Z in E with $\pi_A(Z) = Y$. Then

$$A^{-1}(X) = \pi_A^{-1} A_1^{-1}(X) = \pi_A^{-1}(D + Y) = \pi_A^{-1}(D) + Z.$$

Now $\text{Ker } A$ is a finite dimensional, hence so is $\pi_A^{-1}(D)$ so that $\pi_A^{-1}(D) + Z$ is a local compactoid.

(β) \implies (α). $A^{-1}\{0\}$ is a local compactoid and is also a linear subspace so its dimension is finite. To prove that $\text{Im } A = \text{Im } A_1$ is closed, let $x_1, x_2, \dots \in E/\text{Ker } A$ be such that $\lim_{n \rightarrow \infty} A_1 x_n = y \in F$. As $A_1 x_1, A_1 x_2, \dots$ is Cauchy, $\{A_1 x_1, A_1 x_2, \dots\}$ is a compactoid. By (β), $A^{-1}\{A_1 x_1, A_1 x_2, \dots\}$ is a local compactoid hence so is $\{x_1, x_2, \dots\} \subset \pi_A A^{-1}\{A_1 x_1, A_1 x_2, \dots\}$, and is $X := \overline{\text{co}}\{x_1, x_2, \dots\}$. The norm $x \mapsto \|A_1 x\|$ ($x \in E/\text{Ker } A$) induces a topology on $E/\text{Ker } A$ weaker than the initial topology. Then these topologies coincide on X ([13], Theorems 9, 10). Thus $\lim_{n, m \rightarrow \infty} A_1(x_n - x_m) = 0$ implies $\lim_{n, m \rightarrow \infty} (x_n - x_m) = 0$. By completeness $x := \lim_{n \rightarrow \infty} x_n$ exists and $A_1 x = y$ so $\text{Im } A_1$ is closed.

COROLLARY 3.2. *Let $B \in \mathcal{L}(E, F)$, $A \in \mathcal{L}(F, G)$.*

(i) If A, B are semi-Fredholm then so is AB .

(ii) If AB is semi-Fredholm then so is B .

Proof. Statement (i) is a direct consequence of the previous theorem. To prove (ii), let X be a local compactoid in F . Then $B^{-1}(X) \subset B^{-1}A^{-1}(AX) = (AB)^{-1}(AX)$. Now AX is a local compactoid and, by Theorem 3.1, so is $(AB)^{-1}(AX)$ and is $B^{-1}(X)$.

In the remaining part of § 3 we shall connect the notions ‘compact’ and ‘semi-Fredholm’ and shall see that they are - in a sense - quite opposite.

COROLLARY 3.3. *Let $T, A \in \mathcal{L}(E, F)$. If T is compact and A is semi-Fredholm then $T + A$ is semi-Fredholm.*

Proof. Let $X \subset F$ be an absolutely convex local compactoid; we prove that $(T + A)^{-1}(X)$ is a local compactoid i.e. ([10], Lemma 6.1) that its intersection with B_E is a compactoid. We have

$$\begin{aligned} (T + A)^{-1}(X) \cap B_E &= \{x \in B_E, Tx + Ax \in X\} \\ &\subset \{x \in B_E : Ax \in TB_E + X\} = B_E \cap A^{-1}(TB_E + X). \end{aligned}$$

As TB_E is a compactoid and A is semi-Fredholm, $A^{-1}(TB_E + X)$ is a local compactoid so $B_E \cap A^{-1}(TB_E + X)$ is a compactoid.

THEOREM 3.4. ([8], Theorem 2.4). *Let $A \in \mathcal{L}(E, F)$. The following are equivalent.*

(α) A is semi-Fredholm.

(β) For each closed subspace D of E the restriction $A|D$ is semi-Fredholm.

(γ) For each closed infinite dimensional subspace D of E the restriction $A|D$ is not compact.

(δ) For each closed infinite dimensional subspace D of E , D of countable type, the restriction $A|D$ is not compact.

(ε) For each $t \in (0, 1)$ and each t -orthogonal sequence e_1, e_2, \dots in E with $\inf_n \|e_n\| > 0$ we have $\liminf_{n \rightarrow \infty} \|Ae_n\| > 0$.

Proof. (α) \implies (β). If $X \subset F$ is a local compactoid then, by Theorem 3.1, $A^{-1}(X)$ is one and so is $A^{-1}(X) \cap D$. Again by Theorem 3.1 the restriction $A|D$ is semi-Fredholm.

(β) \implies (γ). Let D be a closed subspace of E such that $A|D$ is compact. From (β) we obtain that AD is closed. By [9], Theorem 4.40 (γ) \implies (α) we have $\dim AD < \infty$. As $(\text{Ker } A) \cap D$ is finite dimensional we must conclude that $\dim D < \infty$.

(γ) \implies (δ). Trivial.

(δ) \implies (ε). Let f_1, f_2, \dots be a subsequence of e_1, e_2, \dots . By (δ) the restriction $A|[\overline{f_1, f_2, \dots}]$ is not compact so not $\lim_{n \rightarrow \infty} Af_n = 0$. It follows that $\liminf_{n \rightarrow \infty} \|Ae_n\| > 0$.

(ε) \implies (α). Suppose A is not semi-Fredholm. By Theorem 3.1 there exists a (closed, absolutely convex) local compactoid $X \subset F$ such that $A^{-1}(X)$ is not a local compactoid

in E . Thus ([9], Theorem 6.7) there exists, for some $t \in (0, 1)$, a t -orthogonal sequence e_1, e_2, \dots in $A^{-1}(X)$ with $\inf_n \|e_n\| > 0$. It is easily seen that the restriction of A to $D := \overline{[e_1, e_2, \dots]}$ is compact. But then D has (by Theorem 2.3) a t -orthogonal base f_1, f_2, \dots with $\inf_n \|f_n\| > 0$ and $\lim_{n \rightarrow \infty} Af_n = 0$ which conflicts (ε) .

THEOREM 3.5. *Let $T \in \mathcal{L}(E, F)$. Then the following are equivalent.*

- (α) T is compact.
- (β) For each closed subspace D of E the restriction $T|D$ is compact.
- (γ) For each closed infinite dimensional subspace D of E the restriction $T|D$ is not semi-Fredholm.
- (δ) For each closed infinite dimensional subspace D of E , D of countable type, the restriction $T|D$ is not semi-Fredholm.
- (ε) For each $t \in (0, 1)$ and each closed infinite dimensional subspace D of E there exists a t -orthogonal sequence e_1, e_2, \dots in D with $\inf_n \|e_n\| > 0$ and $\liminf_{n \rightarrow \infty} \|Te_n\| = 0$.

Proof. (α) \implies (β) is trivial. It is easy to see that (β) \implies (γ) follows from Theorem 3.4 (α) \implies (γ). The implication (γ) \implies (δ) is trivial whereas (α) \implies (ε) follows from Theorem 2.3. To prove (ε) \implies (α) and (δ) \implies (α) first observe that either (ε) or (δ) implies the following.

- (*) For every infinite dimensional closed subspace D of E there exists an infinite dimensional closed subspace D_1 of D for which $T|D_1$ is compact.

(In fact, if (ε) holds, choose $D_1 = \overline{[f_1, f_2, \dots]}$ where f_1, f_2, \dots is a subsequence of e_1, e_2, \dots for which $\lim_{n \rightarrow \infty} \|Tf_n\| = 0$. To prove (δ) \implies (*) we may assume that D is of countable type. By Theorem 3.4 (α) \iff (γ) there exists an infinite dimensional closed subspace $D_1 \subset D$ such that $T|D_1$ is compact.)

We now shall complete the proof by showing (*) \implies (α). Suppose T is not compact. Then $\overline{TB_E}$ is not a compactoid so it contains, for some $t \in (0, 1)$, a t -orthogonal sequence y_1, y_2, \dots with $\alpha := \inf_n \|y_n\| > 0$. Choose $z_n \in TB_E$ with $\|z_n - y_n\| \leq \frac{1}{2}\alpha$ for each n . Then $\|z_n\| = \|y_n\|$ for each n and it is not hard to see that z_1, z_2, \dots are also t -orthogonal. Choose $x_1, x_2, \dots \in B_E$ with $Tx_n = z_n$ for each n . For $\lambda_1, \dots, \lambda_n \in K$ we have

$$\begin{aligned} \|T(\sum_{i=1}^n \lambda_i x_i)\| &= \|\sum_{i=1}^n \lambda_i z_i\| \geq t \max_i \|\lambda_i z_i\| \geq t\alpha \max_i |\lambda_i| \\ &\geq t\alpha \max_i |\lambda_i| \|x_i\| \geq t\alpha \|\sum_{i=1}^n \lambda_i x_i\|. \end{aligned}$$

It follows that $T[\overline{[x_1, x_2, \dots]}]$ is a homeomorphism into F , which contradicts (*).

§ 4. CHARACTERIZATIONS OF SEMI-FREDHOLM OPERATORS

In this section we extend [7], Theorem 5 by allowing the base field K to be not spherically complete (Theorem 4.2).

For an absolutely convex subset A in a K -vector space we set

$$A^e := \begin{cases} A & \text{if the valuation of } K \text{ is discrete} \\ \bigcap_{|\lambda|>1} \lambda A & \text{if the valuation of } K \text{ is dense.} \end{cases}$$

LEMMA 4.1. *Let D be a closed subspace of E such that both D and E/D are infinite-dimensional. Let π be the quotient map $E \rightarrow E/D$. Then there exists a closed subspace F of E such that $\pi(F)$ and $\pi(B_F)^e$ are not closed.*

Proof. Choose a $\frac{1}{2}$ -orthogonal sequence z_1, z_2, \dots in E/D with

$$\frac{1}{2} > \|z_1\| > \|z_2\| > \dots, \quad \lim_{n \rightarrow \infty} z_n = 0.$$

Choose e_1, e_2, \dots in E with $\pi(e_n) = z_n$ and $\|e_n\| \leq 2\|z_n\|$ for each $n \in \mathbb{N}$. Then

$$\|z_n\| \leq \|e_n\| \leq 2\|z_n\| < 1 \quad (n \in \mathbb{N})$$

Further, choose a $\frac{1}{2}$ -orthogonal sequence d_1, d_2, \dots in D for which

$$1 \leq \|d_n\| \leq M \quad (n \in \mathbb{N})$$

for some $M \in \mathbb{R}$. Now set

$$f_n := e_n + d_n \quad (n \in \mathbb{N})$$

and let F be the Banach space generated by $\{f_1, f_2, \dots\}$. We first prove that f_1, f_2, \dots is a $\frac{1}{8}$ -orthogonal base for F . In fact, let $\lambda_1, \dots, \lambda_n \in K$. We have

$$\left\| \sum_{i=1}^n \lambda_i f_i \right\| \geq \left\| \sum_{i=1}^n \pi(\lambda_i f_i) \right\| = \left\| \sum_{i=1}^n \lambda_i z_i \right\| \geq \frac{1}{2} \max_i \|\lambda_i z_i\| \geq \frac{1}{4} \max_i \|\lambda_i e_i\|.$$

So certainly

$$\left\| \sum_{i=1}^n \lambda_i f_i \right\| = \left\| \sum_{i=1}^n \lambda_i e_i + \sum_{i=1}^n \lambda_i d_i \right\| \geq \frac{1}{4} \left\| \sum_{i=1}^n \lambda_i e_i \right\|.$$

But then also

$$\left\| \sum_{i=1}^n \lambda_i f_i \right\| \geq \frac{1}{4} \left\| \sum_{i=1}^n \lambda_i d_i \right\| \geq \frac{1}{8} \max_i \|\lambda_i d_i\| = \frac{1}{8} \max_i \|\lambda_i f_i\|,$$

(since $\|e_i\| < 1 \leq \|d_i\|$ we have $\|d_i\| = \|e_i + d_i\| = \|f_i\|$) which shows that, indeed, f_1, f_2, \dots is an $\frac{1}{8}$ -orthogonal base of F .

By observing that $1 \leq \|f_n\| \leq M$ for each n we find

$$F = \left\{ \sum_{i=1}^{\infty} \lambda_i f_i : \lambda_i \in K, \lim_{i \rightarrow \infty} \lambda_i = 0 \right\}.$$

Hence,

$$\pi(F) = \left\{ \sum_{i=1}^{\infty} \lambda_i z_i : \lambda_i \in K, \lim_{i \rightarrow \infty} \lambda_i = 0 \right\}.$$

We see that $\sum_{i=1}^{\infty} z_i = \lim_{n \rightarrow \infty} \sum_{i=1}^n z_i \in \overline{\pi(F)}$. But, as the system z_1, z_2, \dots is $\frac{1}{2}$ -orthogonal we have $\sum_{i=1}^{\infty} z_i \notin \pi(F)$. Thus, $\pi(F)$ is not closed.

Similarly, by equivalence of the norms $\|\cdot\|$ and $\sum \lambda_i f_i \mapsto \max_i |\lambda_i|$ on F there exist nonzero $\mu, \nu \in K$ for which

$$\nu B_F \subset \left\{ \sum_{i=1}^{\infty} \lambda_i f_i : \lambda_i \in K, |\lambda_i| \leq |\mu|, \lim_{i \rightarrow \infty} \lambda_i = 0 \right\} \subset B_F.$$

Hence,

$$(*) \quad \pi(\nu B_F) \subset \left\{ \sum_{i=1}^{\infty} \lambda_i z_i : \lambda_i \in K, |\lambda_i| \leq |\mu|, \lim_{i \rightarrow \infty} \lambda_i = 0 \right\} \subset \pi(B_F).$$

We see that $z := \mu \sum_{i=1}^{\infty} z_i = \lim_{n \rightarrow \infty} \sum_{i=1}^n \mu z_i \in \overline{\pi(B_F)} \subset \overline{\pi(B_F)^e}$. If $z \in [\pi(B_F)]$ then some nonzero multiple of z is in $\pi(\nu B_F)$ and by $(*)$ and $\frac{1}{2}$ -orthogonality of z_1, z_2, \dots we must have $\mu = 0$, a contradiction. So $z \notin [\pi(B_F)] \supset \pi(B_F)^e$.

Remark. From the non-closedness of $\pi(B_F)^e$ it follows easily that also $\pi(B_F)$ is not closed.

THEOREM 4.2. (Compare [7], Theorem 5.) *Let $A \in \mathcal{L}(E, F)$. The following are equivalent.*

- (α) *A is semi-Fredholm or has finite rank.*
- (β) *If $X \subset E$ is closed and absolutely convex then $(AX)^e$ is closed.*
- (γ) *If $D \subset E$ is a closed subspace then AD is closed.*
- (δ) *If $X \subset E$ is closed, absolutely convex, bounded then $(AX)^e$ is closed.*
- (ϵ) *If $D \subset E$ is a closed subspace then $A(B_D)^e$ is closed.*
- (η) *If $D \subset E$ is a closed subspace then $A(B_D)$ is closed.*

If K is spherically complete then the conditions (α) – (η) of above are equivalent to

$(\beta)'$ If $X \subset E$ is closed and absolutely convex then AX is closed.

$(\delta)'$ If $X \subset E$ is closed, absolutely convex, bounded then AX is closed.

Remarks.

1. For a linear subspace S of F we obviously have $S = S^e$. So, (γ) is equivalent to: If $D \subset E$ is a closed subspace then $(AD)^e$ is closed.
2. If K is not spherically complete neither $(\beta)'$ nor $(\delta)'$ is equivalent to $(\alpha) - (\eta)$. In fact, by [10], Example 6.25 there exists an absolutely convex closed compactoid X in c_0 and an $a \in c_0$ such that $X + Ka$ is not closed. Hence if A is the quotient map $c_0 \rightarrow c_0/Ka$ then A is semi-Fredholm while AX is not closed.

Proof of Theorem 4.2.

$(\alpha) \implies (\beta)$ and $(\alpha) \implies (\beta)'$. If A has finite rank then obviously (β) and $(\beta)'$ are true, so suppose that A is semi-Fredholm. Decompose A as in §3:

$$\begin{array}{ccc} E & \xrightarrow{A} & F \\ & \searrow \pi_A \quad \nearrow A_1 & \\ & E/\text{Ker } A & \end{array}$$

If K is spherically complete then, by finite dimensionality, $\text{Ker } A$ is c -compact so $X + \text{Ker } A$ is closed in E implying that $\pi_A(X)$ is closed in $E/\text{Ker } A$. As A_1 is a homeomorphism into F we have $AX = A_1\pi_A(X)$ is closed in F and we obtain $(\beta)'$. If K is not spherically complete its valuation is dense. Let $\lambda \in K$, $|\lambda| > 1$; we prove that $\overline{\pi_A(X)} \subset \lambda\pi_A(X)$. (Then $\overline{AX} \subset A_1\overline{\pi_A(X)} \subset \lambda AX$ for each $|\lambda| > 1$ i.e. $\overline{AX} \subset (AX)^e$ and (β) follows easily.) By finite dimensionality, $\text{Ker } A$ is a complete metrizable local compactoid, so by [14], Theorem 1.4 we have $\overline{X + \text{Ker } A} \subset \lambda(X + \text{Ker } A)$. Now $\overline{X + \text{Ker } A}$ is closed, absolutely convex and contains $\text{Ker } A = \text{Ker } \pi_A$. As π_A is a quotient map we have that $\pi_A(\overline{X + \text{Ker } A})$ is closed so

$$\overline{\pi_A(X)} \subset \pi_A(\overline{X + \text{Ker } A}) \subset \pi_A(\lambda X + \text{Ker } A) = \pi_A(\lambda X) = \lambda\pi_A(X).$$

$(\beta) \implies (\gamma)$. Trivial.

$(\gamma) \implies (\alpha)$. Suppose A is not of finite rank. From (γ) it follows that $\text{Im } A$ is closed. Suppose $\text{Ker } A$ is infinite dimensional. By Lemma 4.1 we then would have a closed subspace D of E such that $\pi_A(D)$ is not closed in $E/\text{Ker } A$. But then $AD = A_1\pi_A D$ is not closed in F , a contradiction.

$(\alpha) \implies (\eta)$. If A has finite rank then (η) is true so suppose A is semi-Fredholm. Then, by Theorem 3.4, $A|D$ is semi-Fredholm so to prove (η) we may assume $D = E$. It

suffices to show (see the above diagram) that $\pi(B_E)$ is closed. But this is evident since $\pi(B_E)$ is open, absolutely convex, hence closed.

The implications $(\eta) \implies (\varepsilon)$, $(\beta) \implies (\delta) \implies (\varepsilon)$ and $(\beta)' \implies (\delta)' \implies (\eta)$ are trivial. So to complete the proof we shall settle the implication $(\varepsilon) \implies (\alpha)$. We first consider two special cases.

(i) *Assume that A is injective.* Suppose A were not semi-Fredholm. Then by Theorem 3.4 we could find a $t \in (0, 1)$ and a t -orthogonal sequence e_1, e_2, \dots with $\inf_n \|e_n\| > 0$ and $\lim_{n \rightarrow \infty} Ae_n = 0$. We may assume that $\sup_n \|e_n\| < \infty$. Set

$$D_n = \overline{[e_n, e_{n+1}, \dots]}$$

and let B_n be the closed unit ball of D_n . By equivalence of the norms $\|\cdot\|$ and $\sum_{i=1}^{\infty} \lambda_i e_i \mapsto \max |\lambda_i|$ on D_1 there is a $\mu \in K$ such that

$$\overline{\text{co}}\{e_n, e_{n+1}, \dots\} \subset \mu B_n \quad (n \in \mathbb{N})$$

As $\mu(AB_n)^c$ is closed we have for each $n \in \mathbb{N}$

$$\begin{aligned} \overline{\text{co}}\{Ae_n, Ae_{n+1}, \dots\} &\subset \overline{A(\overline{\text{co}}\{e_n, e_{n+1}, \dots\})} \subset \overline{\mu(AB_n)} \\ &\subset \mu(AB_n)^c \subset AD_n. \end{aligned}$$

Set $s := \sum_{i=1}^{\infty} Ae_i$. Then $s \in \overline{\text{co}}\{Ae_1, Ae_2, \dots\} \subset AD_1$, so there exists a (unique) $u \in D_1$ with $s = Au$. Since

$$s \in Ae_1 + \dots + Ae_n + \overline{\text{co}}\{Ae_n, Ae_{n+1}, \dots\} \subset A(e_1 + \dots + e_n) + AD_{n+1}$$

for each $n \in \mathbb{N}$, we can find a $d_{n+1} \in D_{n+1}$ such that $Au = A(e_1 + \dots + e_n + d_{n+1})$. By injectivity of A

$$(*) \quad u = e_1 + \dots + e_n + d_{n+1}.$$

On the other hand, e_1, e_2, \dots is a base of D_1 , so $u = \sum_{i=1}^{\infty} \xi_i e_i$ where $\xi_i \in K$, $\lim_{i \rightarrow \infty} \xi_i = 0$. But $(*)$ yields $\xi_i = 1$ for each i , a contradiction.

(ii) *Suppose A is surjective.* To prove (α) we may assume that F is infinite dimensional and that A is a quotient map $E \rightarrow E/D \simeq F$. If D were infinite dimensional we would have (Lemma 4.1) a closed subspace F of E such that $A(B_F)^c$ is not closed contradicting (ε) .

(iii) *Now let $A \in \mathcal{L}(E, F)$ satisfy (ε) .* As above we decompose A

$$\begin{array}{ccc}
E & \xrightarrow{A} & F \\
\pi_A \searrow & & \nearrow A_1 \\
& E/\text{Ker } A &
\end{array}$$

We shall prove that

- (1) if D is a closed subspace of E then $\pi_A(B_D)^e$ is closed,
 - (2) if D is a closed subspace of $E/\text{Ker } A$ then $A_1(B_D)^e$ is closed.
- (Together with (i), (ii) above this will complete the proof.)

Proof of (1). Let $\lambda \in K$, $|\lambda| > 1$ if the valuation of K is dense, $\lambda = 1$ otherwise. We have $\overline{A_1 \circ \pi_A(B_D)} \subset \lambda A_1 \circ \pi_A(B_D)$, so by injectivity of A_1 :

$$\overline{\pi_A(B_D)} \subset A_1^{-1}(\overline{A_1 \circ \pi_A(B_D)}) \subset \lambda A_1^{-1} A_1(\pi_A(B_D)) = \lambda \pi_A(B_D).$$

It follows that $\pi_A(B_D)^e$ is closed.

Proof of (2). We consider two possibilities.

- (a) The valuation of K is dense. Set $B_D^- := \{x \in D : \|x\| < 1\}$. We have $(B_D^-)^e = B_D$ and, since π_A is a quotient map,

$$B_D \supset \pi_A(B_{\pi_A^{-1}(D)}) \supset B_D^-.$$

By using the injectivity of A_1 for the second equality we obtain

$$A_1(B_D) = A_1((B_D^-)^e) = A_1(B_D^-)^e \subset (A_1(\pi_A(B_{\pi^{-1}(D)})))^e = A(B_{\pi^{-1}(D)})^e.$$

As the latter set is closed we have

$$\overline{A_1(B_D)} \subset (A_1 \pi(B_{\pi^{-1}(D)}))^e \subset A_1(B_D)^e.$$

- (b) The valuation of K is discrete. From (ii) and (1) above it follows that $\text{Ker } A$ is finite dimensional, hence spherically complete. Thus, $\inf\{\|x - y\| : y \in \text{Ker } A\}$ is a minimum for each $x \in E$ so that $\pi_A(B_{\pi_A^{-1}(D)}) = B_D$ and $A_1 B_D = A_1 \pi_A(B_{\pi^{-1}(D)}) = A B_{\pi^{-1}(D)}$. As the latter set is closed, so is the former.

§5. COMPACT OPERATORS $E \rightarrow E$

In this section we prove the non-Archimedean counterparts of the classical Riesz Theory on compact operators thus generalizing the results of [2], [6]. We shall apply (mainly in §6) many properties of the Volume Function of Van Rooij, defined by

$$\text{Vol}(x_1, x_2, \dots, x_n) = \prod_{i=1}^n \text{dist}(x_i, [x_j : j < i])$$

for $x_1, \dots, x_n \in E$. These properties can be found in [10], Chapters I, VI.

THEOREM 5.1. *Let $T \in \mathcal{L}(E)$ be compact, let $\lambda \in K$, $\lambda \neq 0$.*

(i) *Ker $(T - \lambda I)$ is finite dimensional.*

(ii) *Im $(T - \lambda I)$ is closed and is of finite codimension.*

Proof. By Corollary 3.3, $T - \lambda I$ is semi-Fredholm so we have (i) and the first part of (ii). To prove that Im $(T - \lambda I)$ has finite codimension we may assume $\lambda = 1$. Let D_1 be a space in E that is $\frac{1}{2}$ -orthogonal to $D_2 := \text{Im}(T - I)$. Then we have for $d \in D_1$:

$$\|d\| \leq 2 \text{dist}(d, D_2) = 2 \text{dist}(Td, D_2) \leq 2\|Td\|$$

implying that $T|_{D_1}$ is a homeomorphism. By compactness, $\dim D_1 < \infty$. It follows easily that D_2 has finite codimension.

LEMMA 5.2. *Let $\{x_1, x_2, \dots\} \subset E$ be a compactoid. Then*

$$\lim_{n \rightarrow \infty} \text{dist}(x_n, \overline{[x_{n+1}, x_{n+2}, \dots]}) = 0.$$

Proof. Suppose the conclusion were not true. Then there is a $\delta > 0$ and a subsequence x_{n_1}, x_{n_2}, \dots such that for each i

$$\text{dist}(x_{n_i}, \overline{[x_m : m > n_i]}) \geq \delta.$$

Then certainly, with $y_i := x_{n_i}$

$$\text{dist}(y_n, \overline{[y_m : m > n]}) \geq \delta \quad (n \in \mathbb{N}).$$

Now let $k \in \mathbb{N}$. We have

$$\begin{aligned} \delta &\leq \text{dist}(y_1, \overline{[y_m : m > 1]}) \leq \text{dist}(y_1, [y_2, \dots, y_k]) \\ \delta &\leq \text{dist}(y_2, \overline{[y_m : m > 2]}) \leq \text{dist}(y_2, [y_3, \dots, y_k]) \\ &\vdots \\ \delta &\leq \text{dist}(y_k, [y_m : m > k]) \leq \|y_k\|. \end{aligned}$$

Multiplication yields

$$\delta^k \leq \text{Vol}(y_1, \dots, y_k)$$

so that $\sqrt[k]{\text{Vol}(y_1, \dots, y_k)} \geq \delta$.

But, since $\{y_1, y_2, \dots\}$ is a compactoid we must have $\lim_{k \rightarrow \infty} \sqrt[k]{\text{Vol}(y_1, \dots, y_k)} = 0$ ([10], Corollary 6.10), a contradiction.

PROPOSITION 5.3. *Let $T \in \mathcal{L}(E)$ be compact, let $\lambda \in K$, $\lambda \neq 0$. Then the sequence $E \supset (T - \lambda I)E \supset (T - \lambda I)^2 E \supset \dots$ becomes stationary.*

Proof. We may assume $\lambda = 1$. Suppose

$$E \supset \underset{\neq}{SE} \supset \underset{\neq}{S^2 E} \supset \dots$$

where $S := T - I$. As $S^n E$ is closed for each n (Theorem 5.1) we can find a bounded sequence e_0, e_1, \dots where $e_n \in S^n E$ and $\text{dist}(e_n, S^{n+1} E) \geq \frac{1}{2}$ for each n . The set $\{Te_1, Te_2, \dots\}$ is a compactoid so, by the previous lemma, $\lim_{n \rightarrow \infty} \text{dist}(Te_n, \overline{Te_{n+1}, Te_{n+2}, \dots}) = 0$. As $Te_m \in S^{n+1} E$ for each $m > n$ we certainly have $\lim_{n \rightarrow \infty} \text{dist}(Te_n, S^{n+1} E) = 0$. But $Te_n - e_n = Se_n \in S^{n+1} E$ so $\lim_{n \rightarrow \infty} \text{dist}(e_n, S^{n+1} E) = 0$, a contradiction.

In a similar way we have:

PROPOSITION 5.4. *Let $T \in \mathcal{L}(E)$ be compact, let $\lambda \in K$, $\lambda \neq 0$. Then the sequence $0 \subset \text{Ker}(T - \lambda I) \subset \text{Ker}(T - \lambda I)^2 \subset \dots$ becomes stationary.*

Proof. Again, assume $\lambda = 1$. Suppose (setting $S := T - I$)

$$\{0\} = \text{Ker } S^0 \subsetneq \underset{\neq}{\text{Ker } S} \subsetneq \underset{\neq}{\text{Ker } S^2} \subsetneq \dots$$

We can find a bounded sequence e_1, e_2, \dots where $e_n \in \text{Ker } S^n$ and $\text{dist}(e_n, \text{Ker } S^{n-1}) \geq \frac{1}{2}$ for each n . The set $\{Te_1, Te_2, \dots\}$ is a compactoid so by [10], Corollary 6.11, $\lim_{n \rightarrow \infty} \text{dist}(Te_n, [Te_1, \dots, Te_{n-1}]) = 0$. As Te_1, \dots, Te_{n-1} are in $\text{Ker } S^{n-1}$ we have $\lim_{n \rightarrow \infty} \text{dist}(Te_n, \text{Ker } S^{n-1}) = 0$. Now $S^{n-1}(Te_n - e_n) = S^n e_n = 0$, so $Te_n - e_n \in \text{Ker } S^{n-1}$. We find $\lim_{n \rightarrow \infty} \text{dist}(e_n, \text{Ker } S^{n-1}) = 0$, a contradiction.

COROLLARY 5.5. *Let $T \in \mathcal{L}(E)$ be compact, let $\lambda \in K$, $\lambda \neq 0$.*

- (i) *If $T - \lambda I$ is injective it is surjective.*
- (ii) *If $T - \lambda I$ is surjective it is injective.*

Proof.

- (i) By Proposition 5.3, $(T - \lambda I)^m E = (T - \lambda I)^{m+1} E$ for some m . By injectivity of $(T - \lambda I)^m$, $E = (T - \lambda I)E$, i.e. $T - \lambda I$ is surjective.

(ii) Similar. (Use Proposition 5.4.)

For an $A \in \mathcal{L}(E)$ we define its *spectrum* as

$$\sigma(A) = \{\lambda \in K : A - \lambda I \text{ is not invertible}\}.$$

By imitating elementary classical techniques one can show easily that $\sigma(A)$ is a closed and bounded subset of K (which is, in general, not compact) and that for each $\lambda \in \sigma(A)$

$$|\lambda| \leq \inf_n \|A^n\|^{\frac{1}{n}}.$$

Even when K is algebraically closed we may have operators $A \in \mathcal{L}(E)$ for which $\sup\{|\lambda| : \lambda \in \sigma(A)\} < \inf_n \|A^n\|^{\frac{1}{n}}$. (Let E be a complete valued field extension $L \supset K$, considered as a K -Banach space, choose $a \in L \setminus K$ and set $Ax = ax$ ($x \in L$). Then $\sigma(A) = \emptyset$ and $\inf_n \|A^n\|^{\frac{1}{n}} = \|A\| = |a|$.)

In the remaining part of §5 we study $\sigma(T)$ for compact operators T on Banach spaces over arbitrary K , whereas in §6 we shall prove the equality $\max\{|\lambda| : \lambda \in \sigma(T)\} = \inf_n \|T^n\|^{\frac{1}{n}}$ for compact operators on a Banach space over an algebraically closed K .

THEOREM 5.6. *Let $T \in \mathcal{L}(E)$ be compact.*

- (i) *If E is infinite-dimensional then $0 \in \sigma(T)$.*
- (ii) *If $\lambda \in \sigma(T)$, $\lambda \neq 0$ then λ is an eigenvalue of T .*
- (iii) *If $\lambda_1, \lambda_2, \dots \in \sigma(T)$ are distinct then $\lim_{n \rightarrow \infty} \lambda_n = 0$.*
- (iv) *$\sigma(T)$ is compact.*

Proof. (i) This follows for example, from Theorem 3.5 $(\alpha) \implies (\gamma)$, whereas (ii) follows from Corollary 5.5. As (iv) follows easily from (iii) it remains to show (iii). To this end we may suppose that all λ_n are $\neq 0$. For each n , choose an eigenvector e_n for λ_n and set $D_n = [e_1, \dots, e_n]$. Then $\dim D_n = n$. Inductively we can construct a bounded sequence x_1, x_2, \dots for which

$$x_n \in D_n \quad , \quad \text{dist}(x_{n+1}, D_n) \geq \frac{1}{2} \quad (n \in \mathbb{N})$$

For each n we have $Tx_{n+1} - \lambda_{n+1}x_{n+1} \in D_n$ (*Proof.* Set $x_{n+1} = \xi_1 e_1 + \dots + \xi_{n+1} e_{n+1}$. Then $Tx_{n+1} - \lambda_{n+1}x_{n+1} = \sum_{i=1}^{n+1} (\lambda_i - \lambda_{n+1}) \xi_i e_i \in [e_1, \dots, e_n] = D_n$). Since Tx_1, Tx_2, \dots is a compactoid we have by [10], Corollary 6.11 that $\lim_{n \rightarrow \infty} \text{dist}(Tx_{n+1}, [Tx_1, \dots, Tx_n]) = 0$. But

$$[Tx_1, \dots, Tx_n] \subset D_n$$

$$Tx_{n+1} - \lambda_{n+1}x_{n+1} \in D_n$$

so

$$\lim_{n \rightarrow \infty} \text{dist}(\lambda_{n+1}x_{n+1}, D_n) = 0.$$

On the other hand,

$$\text{dist}(\lambda_{n+1}x_{n+1}, D_n) = |\lambda_{n+1}| \text{dist}(x_{n+1}, D_n) \geq \frac{1}{2}|\lambda_{n+1}|.$$

It follows that $\lim_{n \rightarrow \infty} \lambda_n = 0$.

Finally we shall formulate a p -adic version of Fredholm's alternative.

PROPOSITION 5.7. *Let $T \in \mathcal{L}(E, F)$ and let $T' : \mathcal{L}(F', E')$ be its adjoint.*

- (i) $\|T'\| \leq \|T\|$.
- (ii) *If T has finite rank then so has T' .*
- (iii) *If T is compact then so is T' .*

Proof. (i) and (ii) are obvious. To prove (iii), Let $n \mapsto T_n \in \mathcal{L}(E, F)$ be a sequence of finite rank operators converging to T ([9], Theorem 4.39). Then each T'_n has finite rank and $\|T' - T'_n\| \leq \|T - T_n\| \rightarrow 0$, so T' is compact.

To obtain further conclusions we shall assume that our space is polar in the sense of [12]. (It is easily seen that a Banach space is polar if and only if the natural map $j_E : E \rightarrow E''$ is a linear homeomorphism into E'').

PROPOSITION 5.8. *Let E be a polar space, let $T \in \mathcal{L}(E)$.*

- (i) *If T' is compact then so is T .*
- (ii) *If T is compact then $\sigma(T) = \sigma(T')$.*

Proof (i). From Proposition 5.7 (iii) we obtain that T'' is compact. The commutative diagram

$$\begin{array}{ccc} E & \xrightarrow{T} & E \\ j_E \downarrow & & \downarrow j_E \\ E'' & \xrightarrow{T''} & E'' \end{array}$$

tells us that $j_{E''}T = T'' \circ j_E$ is compact. As j_E is a homeomorphism, T must be compact.
(ii). If $\lambda \in K$, $\lambda \notin \sigma(T)$ then $T - \lambda I$ is invertible hence so is $T' - \lambda I'$ i.e. $\lambda \notin \sigma(T')$. Thus, $\sigma(T') \subset \sigma(T)$, and by the same token, $\sigma(T'') \subset \sigma(T') \subset \sigma(T)$. We complete the proof by showing $\sigma(T'') \supset \sigma(T)$. So, let $\lambda \in K$, $\lambda \notin \sigma(T'')$. Then $T'' - \lambda I''$ is invertible, so from the commutative diagram

$$\begin{array}{ccc}
E & \xrightarrow{T-\lambda I} & E \\
j_E \downarrow & & \downarrow j_E \\
E'' & \xrightarrow{T''-\lambda I''} & E''
\end{array}$$

and the injectivity of j_E we obtain that $T - \lambda I$ is injective. By Corollary 5.5 we have surjectivity of $T - \lambda I$ hence $\lambda \notin \sigma(T)$.

As a corollary we now formulate:

THEOREM 5.9. (FREDHOLM'S ALTERNATIVE) *Let E be a polar Banach space, let $T \in \mathcal{L}(E)$ be compact. Let $\lambda \in K$, $\lambda \neq 0$. Then we have precisely one of the following two possibilities (a) or (b).*

- (a) $T - \lambda I$ and $T' - \lambda I$ are invertible.
- (b) λ is an eigenvalue of T and λ is an eigenvalue of T' .

Proof. Proposition 5.8 and Theorem 5.6 (ii).

§6. THE SPECTRAL THEOREM FOR COMPACT OPERATORS

The main result of this section was proved in [4], but we shall use an approach that is more down-to-earth.

THROUGHOUT §6 WE ASSUME THAT K IS ALGEBRAICALLY CLOSED.

We need a few remarks on vector-valued analytic functions.

DEFINITION 6.1. Let X be a K -Banach space, let $\alpha \in K$, $\alpha \neq 0$. Set $B(0, |\alpha|) = \{\lambda \in K : |\lambda| \leq |\alpha|\}$. A function $f : B(0, |\alpha|) \rightarrow X$ is *analytic* if there exist $a_0, a_1, \dots \in X$ such that

$$(*) \quad f(\lambda) = \sum_{n=0}^{\infty} a_n \lambda^n \quad (\lambda \in B(0, |\alpha|))$$

PROPOSITION 6.2. If $f : B(0, |\alpha|) \rightarrow X$ is analytic with expansion $(*)$ then

$$\|f\|_{\infty} := \sup_{\lambda \in B(0, |\alpha|)} \|f(\lambda)\| = \max_n \|a_n\| |\alpha|^n$$

Proof. It suffices to consider the case $\alpha = 1$ ($\lambda \rightarrow f(\lambda\alpha)$ is analytic on $B(0, 1)$). From $(*)$ we obtain, by taking $\lambda = 1$, that $\lim_{n \rightarrow \infty} a_n = 0$ so there is an $m \in \{0, 1, 2, \dots\}$ with $\|a_m\| = \max_i \|a_i\|$. The space $F := \overline{[a_0, a_1, \dots]}$ is of countable type, so letting $0 < \varepsilon < 1$ we can find a $\varphi \in F'$ with $\varphi \neq 0$ and $|\varphi(a_m)| \geq (1 - \varepsilon)\|\varphi\| \|a_m\|$. Now $\varphi \circ f$ is a K -valued analytic function for which the conclusion of 6.2 is well known. Thus, we have

$$\begin{aligned} \|\varphi\| \|f\|_{\infty} &\geq \|\varphi \circ f\|_{\infty} = \sup_{|\lambda| \leq 1} \left| \sum_{n=0}^{\infty} \varphi(a_n) \lambda^n \right| = \max_n |\varphi(a_n)| \\ &\geq |\varphi(a_m)| \geq (1 - \varepsilon)\|\varphi\| \|a_m\| \end{aligned}$$

and we obtain

$$\|f\|_{\infty} \geq (1 - \varepsilon)\|a_m\| = (1 - \varepsilon) \max_n \|a_n\|$$

for each $\varepsilon \in (0, 1)$ yielding

$$\|f\|_{\infty} \geq \max_n \|a_n\|$$

concluding the proof since the opposite inequality is trivial.

COROLLARY 6.3.

- (i) If $f(\lambda) = \sum_{n=0}^{\infty} a_n \lambda^n = \sum_{n=0}^{\infty} b_n \lambda^n$ ($\lambda \in B(0, |\alpha|)$) then $a_n = b_n$ for each n .
- (ii) The set of analytic functions $B(0, |\alpha|) \rightarrow X$ is uniformly closed.

Proof. (i) Follows directly by applying Proposition 6.2 to $\lambda \mapsto \sum_{n=0}^{\infty} (a_n - b_n) \lambda^n$.

(ii) It suffices to consider the case $\alpha = 1$. Proposition 6.2 shows that the map sending an analytic $f : B(0, 1) \rightarrow X$ into the sequence (a_0, a_1, \dots) of its coefficients is a K -linear isometry of the space $H(B(0, 1), X)$ of all analytic functions $B(0, 1) \rightarrow X$ into $c_0(E)$, the space consisting of all null sequences in E . As $c_0(E)$ is complete, so is $H(B(0, 1), X)$, and (ii) follows.

PROPOSITION 6.4. *Let X be a finite-dimensional Banach space. For an $f : B(0, |\alpha|) \rightarrow X$ the following are equivalent.*

(α) *f is analytic.*

(β) *For each $\varphi \in X'$ the function $\varphi \circ f : B(0, |\alpha|) \rightarrow K$ is analytic.*

Proof. (α) \Rightarrow (β) is obvious. Suppose (β). Then for each $\varphi \in X'$ there exist $a_0(\varphi), a_1(\varphi), \dots \in K$ such that

$$\varphi(f(\lambda)) = \sum_{n=0}^{\infty} a_n(\varphi) \lambda^n \quad (\lambda \in B(0, |\alpha|)).$$

For each n the map $\varphi \mapsto a_n(\varphi)$ is in X'' . By reflexivity, there exists a sequence b_0, b_1, \dots in X with $a_n(\varphi) = \varphi(b_n)$ for each $\varphi \in X'$. Then $\varphi(f(\lambda)) = \sum_{n=0}^{\infty} \varphi(b_n) \lambda^n = \varphi(\sum_{n=0}^{\infty} b_n \lambda^n)$ for all $\lambda \in B(0, |\alpha|)$, all $\varphi \in X'$ and it follows that $f(\lambda) = \sum_{n=0}^{\infty} b_n \lambda^n$ ($\lambda \in B(0, |\alpha|)$).

DEFINITION 6.5. Let $A \in \mathcal{L}(E)$. Set

$$\nu(A) = \inf_n \|A^n\|^{\frac{1}{n}} = \lim_n \|A^n\|^{\frac{1}{n}} \quad (\text{see [9], Theorem 6.22}).$$

A is a *spectral operator* if

$$\sup\{|\lambda| : \lambda \in \sigma(A)\} = \nu(A).$$

For $A \in \mathcal{L}(E)$ set

$$U_A = \{\lambda \in K : (I - \lambda A)^{-1} \text{ exists in } \mathcal{L}(E)\}$$

(U_A is open and $0 \in U_A$) and

$$C_A = \{\alpha \in K : B(0, |\beta|) \subset U_A \text{ for some } \beta \in K, |\beta| > |\alpha|\}$$

PROPOSITION 6.6. *For an $A \in \mathcal{L}(E)$ the following are equivalent.*

(α) *A is a spectral operator.*

(β) $(I - \lambda A)^{-1} = \sum_{n=0}^{\infty} \lambda^n A^n$ ($\lambda \in C_A$)

(γ) For each $\alpha \in C_A$, $\alpha \neq 0$ the function $\lambda \mapsto (I - \lambda A)^{-1}$ is analytic on $B(0, |\alpha|)$.

Proof. By uniqueness of coefficients (Corollary 6.3 (i)) and the fact that formula (β) holds for small $|\lambda|$ we may conclude (β) \Leftrightarrow (γ). To prove (α) \Rightarrow (β), let $\lambda \in C_A$. There is a $\beta \in K$, $|\beta| > |\lambda|$ with: $I - \mu A$ is invertible for each $|\mu| \leq |\beta|$ i.e. $A - \nu I$ is invertible for each $|\nu| > |\beta|^{-1}$. Thus for each element $\tau \in \sigma(A)$ we have $|\tau| \leq |\beta|^{-1}$. As A is spectral, $\nu(A) \leq |\beta|^{-1}$ i.e. $\nu(\beta A) \leq 1$, so $\nu(\lambda A) < 1$, and (β) follows. Conversely, suppose (β). Let $\lambda \in K$, $|\lambda| > \sup\{|\tau| : \tau \in \sigma(A)\}$; we prove that $|\lambda| > \nu(A)$. Choose a $\lambda_1 \in K$ with $|\lambda| > |\lambda_1| > \sup\{|\tau| : \tau \in \sigma(A)\}$. Then $I - \mu A$ exists for all $|\mu| \leq |\lambda_1|^{-1}$ and it follows that $\lambda^{-1} \in C_A$. By (β), $\nu(\lambda^{-1} A) < 1$ i.e. $|\lambda| > \nu(A)$.

Remark. It is not hard to see that, for any $A \in \mathcal{L}(E)$, the map $\lambda \mapsto (I - \lambda A)^{-1}$ is C^∞ on U_A .

PROPOSITION 6.7. *Let $A \in \mathcal{L}(E)$. If the algebra generated by A is finite dimensional then A is spectral.*

Proof. Let \mathfrak{A} be the algebra generated by I and A . If Z is an algebra, $\mathfrak{A} \subset Z$, then, by finite dimensionality, invertibility of an element of \mathfrak{A} is the same as invertibility in Z . Thus, we may assume that \mathfrak{A} is isomorphic to the $n \times n$ -matrix algebra for some $n \in \mathbb{N}$. For each $\lambda \in U_A$ the matrix coefficients of $I - \lambda A$ are polynomials (of degree ≤ 1 .) By the Cramer rule, the matrix coefficients of $(I - \lambda A)^{-1}$ are rational functions of $\lambda \in U_A$, hence analytic on each disc $B(0, |\alpha|)$ where $\alpha \in C_A$. Now use Proposition 6.6.

COROLLARY 6.8. *If $A \in \mathcal{L}(E)$ has finite rank then A is spectral.*

LEMMA 6.9. *Let A_1, A_2, \dots be spectral operators and let $A = \lim_{n \rightarrow \infty} A_n$ in the sense of the norm. Suppose that for every $\alpha \in C_A$*

$$M_\alpha := \sup_{|\lambda| \leq |\alpha|} \|(I - \lambda A)^{-1}\| < \infty.$$

Then A is a spectral operator.

Proof. Let $\alpha \in C_A$, $\alpha \neq 0$. We may suppose $\|A - A_n\| < (|\alpha| M_\alpha)^{-1}$ for each n . Let $\lambda \in B(0, |\alpha|)$. Since

$$I - \lambda A_n = (I - \lambda A)(I + (I - \lambda A)^{-1} \lambda (A - A_n))$$

$$\|(I - \lambda A)^{-1} \lambda (A - A_n)\| \leq M_\alpha |\alpha| \|A - A_n\| < 1$$

the operator $I - \lambda A_n$ is invertible for each n and

$$\begin{aligned} \|(I - \lambda A_n)^{-1}\| &\leq \|(I - \lambda A)^{-1}\| \left\| \sum_{k=0}^{\infty} (-1)^k ((I - \lambda A)^{-1} \lambda (A - A_n))^k \right\| \\ &\leq \|(I - \lambda A)^{-1}\| \leq M_\alpha \end{aligned}$$

Now from

$$\begin{aligned} \|(I - \lambda A)^{-1} - (I - \lambda A_n)^{-1}\| &= \|(I - \lambda A_n)^{-1} \lambda (A - A_n) (I - \lambda A)^{-1}\| \\ &\leq M_\alpha^2 |\alpha| \|A - A_n\| \end{aligned}$$

we obtain $\lim_{n \rightarrow \infty} (I - \lambda A_n)^{-1} = (I - \lambda A)^{-1}$ uniformly in $\lambda \in B(0, |\alpha|)$, so $\lambda \mapsto (I - \lambda A)^{-1}$ is analytic on $B(0, |\alpha|)$ by Corollary 6.3, and A is spectral by Proposition 6.6 $(\gamma) \Rightarrow (\alpha)$.

To arrive at the conclusion that a compact operator A is spectral it suffices, by Lemma 6.9, to show that $M_\alpha < \infty$ for each $\alpha \in C_A$. To this end we introduce certain numbers associated to A .

DEFINITION 6.10. Let E be infinite dimensional, let $A \in \mathcal{L}(E)$. For $k \in \mathbb{N}$ set

$$\begin{aligned} \Delta_k(A) &= \sup \left\{ \frac{\text{Vol}(Ax_1, \dots, Ax_k)}{\text{Vol}(x_1, \dots, x_k)} : x_1, \dots, x_k \text{ linearly independent} \right\} \\ \Delta_-(A) &= \liminf_{k \rightarrow \infty} \sqrt[k]{\Delta_k(A)} \\ \Delta_+(A) &= \limsup_{k \rightarrow \infty} \sqrt[k]{\Delta_k(A)}. \end{aligned}$$

It is not difficult to see that, using

$$\frac{\text{Vol}(Ax_1, \dots, Ax_k)}{\text{Vol}(x_1, \dots, x_k)} = \frac{\text{Vol}(Ay_1, \dots, Ay_k)}{\text{Vol}(y_1, \dots, y_k)}$$

as soon as $[x_1, \dots, x_k] = [y_1, \dots, y_k]$ is k -dimensional ([10], Corollary 1.5) we have

$$\begin{aligned} \Delta_k(A) &= \sup \{ \text{Vol}(Ax_1, \dots, Ax_k) : \|x_i\| \leq 1 \text{ for each } i \} \\ \Delta_k(A) &\leq \|A\|^k \\ 0 &\leq \Delta_-(A) \leq \Delta_+(A) \leq \|A\|. \end{aligned}$$

PROPOSITION 6.11. Let $A \in \mathcal{L}(E)$ be compact. Then $\Delta_+(A) = 0$.

Proof. By compactoidity of the image of the unit ball under A and by [10], Corollary 6.10 there exist positive numbers c_1, c_2, \dots tending to 0 such that for all x_1, \dots, x_k in the unit ball

$$\text{Vol}(Ax_1, \dots, Ax_k) \leq c_1 c_2 \dots c_k \quad (k \in \mathbb{N})$$

Then

$$\Delta_k(A) \leq c_1 c_2 \dots c_k \quad (k \in \mathbb{N})$$

so that

$$\Delta_+(A) = \limsup_{k \rightarrow \infty} \sqrt[k]{\Delta_k(A)} \leq \limsup_{k \rightarrow \infty} \sqrt[k]{c_1 \dots c_k} = 0.$$

Finally we prove:

PROPOSITION 6.12. *Let $A \in \mathcal{L}(E)$ be such that*

$$M_\alpha = \sup_{|\lambda| \leq |\alpha|} \|(I - \lambda A)^{-1}\| = \infty$$

for some $\alpha \in C_A$ then $\Delta_-(A) > 0$.

Proof. I. There exists a sequence $\lambda_1, \lambda_2, \dots \in K$ with $|\lambda_n| \leq |\alpha|$ for each n with $\|(I - \lambda_n A)^{-1}\| \rightarrow \infty$. So there exists a sequence y_1, y_2, \dots in E tending to 0 such that for

$$x_n := (I - \lambda_n A)^{-1} y_n$$

we have $\inf_n \|x_n\| > 0$ and $\sup_n \|x_n\| < \infty$. We claim that $\lambda_1, \lambda_2, \dots$ does not have a convergent subsequence. In fact if $\lambda_{n_i} \rightarrow \lambda$ then $|\lambda| \leq |\alpha|$ but $(I - \lambda A)x_{n_i} \rightarrow 0$ implying that $I - \lambda A$ is not invertible. So, by taking a suitable subsequence, we may assume $\inf_{n \neq m} |\lambda_n - \lambda_m| > 0$. Now the space $D = \overline{[x_1, x_2, \dots]}$ is of countable type, so there exists a norm $\|\cdot\|'$ on D , equivalent to $\|\cdot\|$, with $\|x\|' \in |K|$ for all $x \in D$. By extending this norm to a norm $\|\cdot\|''$ on E , equivalent to $\|\cdot\|$ and by multiplication of x_n by suitable scalars we can arrange that $\|x_n\|'' = 1$ for all n . Since ' $\Delta_-(A) > 0$ ' does not depend on the choice of an equivalent norm we may suppose that $\|\cdot\|'' = \|\cdot\|$ i.e. that $\|x_n\| = 1$ for each n . Finally it is easy to see that without loss of generality we may assume that $\|A\| \leq 1$.

II. From I it follows that we may assume the existence of $\lambda_1, \lambda_2, \dots \in K$ and $x_1, x_2, \dots \in E$ such that

- (i) $\|A\| \leq 1$
- (ii) $|\lambda_n| \leq |\alpha|$ for each n
- (iii) $\rho := \inf_{n \neq m} |\lambda_n - \lambda_m| > 0$
- (iv) $\|x_n\| = 1$ for all n
- (v) $\lim_{n \rightarrow \infty} (x_n - \lambda_n A x_n) = 0$.

We claim that for each $n \in \mathbb{N}$ there exists a positive number $r(n)$ such that

$$\text{Vol}(x_{k+1}, x_{k+2}, \dots, x_{k+n}) \geq r(n)$$

for almost all $k \in \mathbb{N}$.

Proof of the claim. By induction with respect to n . Choose $r(1) = 1$. Suppose $r(1), r(2), \dots, r(n-1)$ have been defined with the right property. Then there is a $k_0 \in \mathbb{N}$ such that

$$\text{Vol}(x_{k+1}, x_{k+2}, \dots, x_{k+n-1}) \geq r(n-1) \quad (k \geq k_0)$$

Choose $0 < \varepsilon < \rho \cdot (r(1) \cdot r(2) \dots r(n-1))^2$ and $k_1 \geq k_0$ such that $\|x_m - \lambda_m A x_m\| < \varepsilon$ for all $m \geq k_1$. We shall prove that for $m \geq k_1$

$$\text{Vol}(x_{m+1}, x_{m+2}, \dots, x_{m+n}) \geq r(n)$$

where

$$r(n) := r(n-1) \cdot |\alpha|^{-1} \rho \cdot r(1) \dots r(n-1).$$

In fact, by the induction hypothesis, we have for $m \geq k_1 \geq k_0$

$$\begin{aligned} \text{Vol}(x_{m+1}, x_{m+2}, \dots, x_{m+n}) &= \text{dist}((x_{m+n}, [x_{m+1}, \dots, x_{m+n-1}])). \text{Vol}(x_{m+1}, \dots, x_{m+n-1}) \\ &\geq \text{dist}(x_{m+n}, [x_{m+1}, \dots, x_{m+n-1}])). r(n-1) \end{aligned}$$

Thus we have to show that for each choice of $\xi_1, \dots, \xi_{n-1} \in K$

$$y := x_{m+n} - (\xi_1 x_{m+1} + \dots + \xi_{n-1} x_{m+n-1})$$

has norm $\geq |\alpha|^{-1} \rho r(1) \dots r(n-1)$. Set $t = r(1) \dots r(n-1)$. As $|\alpha|^{-1} \rho t \leq 1$ we have $\|y\| \geq |\alpha|^{-1} \rho t$ if $\|\xi_1 x_{m+1} + \dots + \xi_{n-1} x_{m+n-1}\| \neq 1$. Thus, we may assume $1 = \|x_{m+n}\| = \|\xi_1 x_{m+1} + \dots + \xi_{n-1} x_{m+n-1}\|$.

We have

$$(1) \quad \|Ay\| = \|Ax_{m+n} - \sum_{i=1}^{n-1} A(\xi_i \cdot x_{m+i})\| \leq \|y\| \quad (\text{since } \|A\| \leq 1)$$

$$(2) \quad \|\lambda_{m+n} y\| = \|\lambda_{m+n} x_{m+n} - \sum_{i=1}^{n-1} \lambda_{m+n} \xi_i x_{m+i}\| \leq |\alpha| \|y\|$$

$$(3) \quad \|Ax_{m+i} - \lambda_{m+i} x_{m+i}\| < \varepsilon \quad (i \in \{1, \dots, n\}).$$

Combination of (1), (2), (3) yields

$$(*) \quad \left\| \sum_{i=1}^{n-1} \xi_i (\lambda_{m+n} - \lambda_{m+i}) x_{m+i} \right\| \leq \varepsilon \vee \varepsilon \max |\xi_i| \vee \|y\| \vee |\alpha| \|y\|$$

By the induction hypotheses, the $x_{m+1}, \dots, x_{m+n-1}$ are t -orthogonal. So $1 \geq \|\sum_{i=1}^{n-1} \xi_i x_{m+i}\| \geq t \max |\xi_i|$ and hence $\max |\xi_i| \leq t^{-1}$. With this information and the fact that $t^{-1} \geq 1$ and $|\alpha| \geq 1$ (since $\|A\| < 1$) $(*)$ becomes

$$(**) \quad \left\| \sum_{i=1}^{n-1} \xi_i (\lambda_{m+n} - \lambda_{m+i}) x_{m+i} \right\| \leq \varepsilon t^{-1} \vee |\alpha| \|y\|$$

On the other hand, we have by the same t -orthogonality

$$\begin{aligned} \left\| \sum_{i=1}^{n-1} \xi_i (\lambda_{m+n} - \lambda_{m+i}) x_{m+i} \right\| &\geq t \max_{1 \leq i < n} |\xi_i| |\lambda_{m+n} - \lambda_{m+i}| \\ &\geq t \rho \max |\xi_i| \geq t \rho \|\xi_1 x_{m+1} + \dots + \xi_{n-1} x_{m+n-1}\| = t \rho. \end{aligned}$$

This, combined with (**) yields

$$t \rho \leq t^{-1} \varepsilon \vee |\alpha| \|y\|$$

By our choice $\varepsilon < \rho t^2$ we therefore must have

$$t \rho \leq |\alpha| \|y\| \quad \text{i.e.} \quad \|y\| \geq |\alpha|^{-1} \rho t,$$

which finishes the proof of the claim.

III. Finally we prove that $\Delta_-(A) > 0$. Let $\lambda_1, \lambda_2, \dots$ and x_1, x_2, \dots be as in Part II. Let $n \in \mathbb{N}$. Choose an ε with $0 < \varepsilon < r(n)$, and choose a $k_0 \in \mathbb{N}$ such that $\text{Vol}(x_{k+1}, x_{k+2}, \dots, x_{k+n}) \geq r(n)$ for all $k \geq k_0$ (this is part II), and also such that $\|x_j - \lambda_j A x_j\| < \varepsilon$ for $j \geq k_0$. By Lemma 6.13 below we then have for $k \geq k_0$

$$(***) \quad \text{Vol}(x_{k+1}, \dots, x_{k+n}) = \text{Vol}(\lambda_{k+1} A x_{k+1}, \dots, \lambda_{k+n} A x_{k+n})$$

(part II has been set up to obtain this equality). By using the properties of the Volume function and the definition of $\Delta_n(A)$ we obtain from (***):

$$\begin{aligned} \text{Vol}(x_{k+1}, \dots, x_{k+n}) &= |\lambda_{k+1} \dots \lambda_{k+n}| \text{Vol}(A x_{k+1}, \dots, A x_{k+n}) \\ &\leq |\lambda_{k+1} \dots \lambda_{k+n}| \Delta_n(A) \text{Vol}(x_{k+1}, \dots, x_{k+n}) \end{aligned}$$

It follows that

$$\Delta_n(A) \geq |\lambda_{k+1} \dots \lambda_{k+n}|^{-1} \geq |\alpha|^{-n}.$$

As n was arbitrary we find

$$\Delta_-(A) = \liminf_{n \rightarrow \infty} \sqrt[n]{\Delta_n(A)} \geq |\alpha| > 0.$$

LEMMA 6.13. *Let $x_1, \dots, x_n \in E$, $\|x_i\| \leq 1$ for each i and $0 < \varepsilon < \text{Vol}(x_1, \dots, x_n)$. If $y_1, \dots, y_n \in E$, $\|y_i - x_i\| < \varepsilon$ for each i then $\text{Vol}(x_1, \dots, x_n) = \text{Vol}(y_1, \dots, y_n)$.*

Proof. Observe that $\varepsilon < 1$ so that $\|y_i\| \leq 1$ for each i . By the symmetry of the Volume function it suffices to prove that $\text{Vol}(x_1, \dots, x_n) = \text{Vol}(y_1, x_2, \dots, x_n)$. We have

$$\begin{aligned} \text{Vol}(x_1, \dots, x_n) &= \text{dist}(x_1, [x_2, \dots, x_n]) \text{Vol}(x_2, \dots, x_n) \\ &\leq \max[\text{dist}(x_1 - y_1, [x_2, \dots, x_n]), \text{dist}(y_1, [x_2, \dots, x_n])] \cdot \text{Vol}(x_2, \dots, x_n) \\ &\leq \max[\varepsilon \text{Vol}(x_2, \dots, x_n), \text{dist}(y_1, [x_2, \dots, x_n]) \cdot \text{Vol}(x_2, \dots, x_n)] \\ &\leq \max(\varepsilon, \text{Vol}(y_1, x_2, \dots, x_n)). \end{aligned}$$

As $\varepsilon < \text{Vol}(x_1, \dots, x_n)$ we find $\text{Vol}(x_1, \dots, x_n) \leq \text{Vol}(y_1, x_2, \dots, x_n)$. To prove the opposite inequality, observe that $\varepsilon < \text{Vol}(y_1, x_2, \dots, x_n)$ so we can repeat the first part of this proof, where y_1 and x_1 are interchanged.

As a corollary we obtain

THEOREM 6.14. (SPECTRAL THEOREM FOR COMPACT OPERATORS) *Let E be a Banach space over an algebraically closed K . Then for each compact operator $A \in \mathcal{L}(E)$*

$$\max\{|\lambda| \mid \lambda \in \sigma(A)\} = \lim_{n \rightarrow \infty} \|A^n\|^{\frac{1}{n}} = \nu(A).$$

Proof. From Propositions 6.11 and 6.12 it follows that $M_\alpha < \infty$ for each $\alpha \in C_A$. Let A_1, A_2, \dots be operators of finite rank with $\lim_{n \rightarrow \infty} \|A - A_n\| = 0$. Each A_n is spectral by Proposition 6.7. Then A is spectral by Lemma 6.9.

Remark. The conclusion of Theorem 6.14 also holds if $A \in \mathcal{L}(E)$ and A^n is compact for some $n \in \mathbb{N}$ ($\sigma(A)^n = \sigma(A^n)$ and $\nu(A^n) = \nu(A)^n$).

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